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Geometrical methods in loop calculations and the three-point function

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A geometrical way to calculate N-point Feynman diagrams is reviewed. As an example, the dimensionally-regulated three-point function is considered, including all orders of its ε-expansion. Analytical continuation to other regions of the kinematical variables is discussed.

1. INTRODUCTION

The analytical structure of the results for N-point Feynman diagrams can be better understood if one employs a geometrical interpretation of kinematic invariants and other quantities. For example, the singularities of the general three-point function can be described pictorially through a tetrahedron constructed out of the external momenta and internal masses (see Fig. 1a). This method can be used to derive Landau equations defining the positions of possible singularities [1] (see also in [2]).

In Ref. [3] it was demonstrated how such geometrical ideas could be used for an analytical calculation of one-loop N-point diagrams. For example, in the three-point case in n dimensions, the result can be expressed in terms of an integral over a spherical (or hyperbolic) triangle, as shown in Fig. 1b, with a weight function $\cos^{3-n}\theta$, where θ is the angular distance between the integration point and the point 0, corresponding to the height of the basic tetrahedron (see in [3]). This weight function equals 1 for $n=3$ (see also in [4]). For $n=4$, one can get another representation [5,6], in terms of an integral over the volume of an asymptotic hyperbolic tetrahedron. Here we will discuss the application of the approach of Refs. [3,7] to the three-point function in any dimension n, as well as its ε-expansion ($n=4-2\varepsilon$) within dimensional regularization [8].

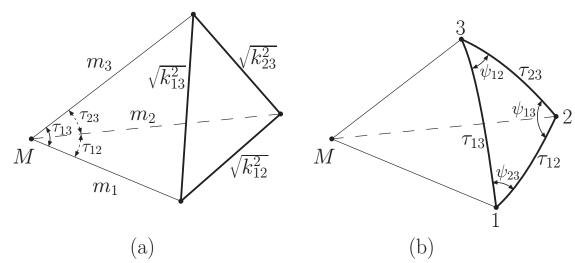

Figure 1. Three-point case: (a) the basic tetrahedron and (b) the solid angle

2. TRIGONOMETRIC STUFF

We will use notations defined in [3], namely:

$$c_{jl} = \cos\tau_{jl} \equiv \frac{m_j^2 + m_l^2 - k_{jl}^2}{2m_j m_l}, \tag{1}$$

$$\cos\tau_{0i} = \frac{m_0}{m_i}, \qquad m_0 = m_1 m_2 m_3 \sqrt{\frac{D^{(3)}}{\Lambda^{(3)}}}, \tag{2}$$

$$\begin{aligned}\Lambda^{(3)} &= -\tfrac{1}{4}\left[(k_{12}^2)^2 + (k_{13}^2)^2 + (k_{23}^2)^2 \right.\\ &\quad \left. -2k_{12}^2 k_{13}^2 - 2k_{12}^2 k_{23}^2 - 2k_{13}^2 k_{23}^2\right] \\ &= -\tfrac{1}{4}\lambda\left(k_{12}^2, k_{13}^2, k_{23}^2\right), \end{aligned} \tag{3}$$

$$\begin{aligned}D^{(3)} &\equiv \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix} \\ &= 1 - c_{12}^2 - c_{13}^2 - c_{23}^2 + 2c_{12}c_{13}c_{23}. \end{aligned} \tag{4}$$

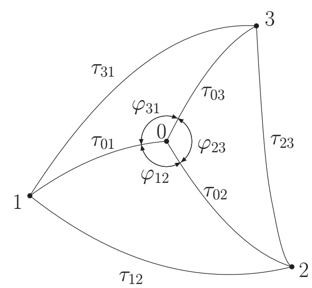

Figure 2. The spherical triangle 123

Assuming that all $|c_{jl}| \leq 1$, $\Lambda^{(3)} > 0$, and $D^{(3)} > 0$, we get spherical triangles. In other cases, we need to use the hyperbolic space. The transition corresponds to the analytic continuation (see below).

Using the approach of Ref. [3], we split the spherical triangle into three smaller ones (see Fig. 2), so that $\varphi_{12} + \varphi_{23} + \varphi_{31} = 2\pi$. Moreover, it is convenient to split each of the resulting triangles into two rectangular ones, as shown in Fig. 3. By definition, $\frac{1}{2}\left(\varphi_{12}^{+} + \varphi_{12}^{-}\right) = \varphi_{12}$, $\frac{1}{2}\left(\tau_{12}^{+} + \tau_{12}^{-}\right) = \tau_{12}$. Let us list useful relations between the sides and angles of the spherical triangle (see also in [3]):

$$\cos\left(\tfrac{1}{2}\tau_{12}^{+}\right) = \frac{\cos\tau_{01}}{\cos\eta_{12}},$$

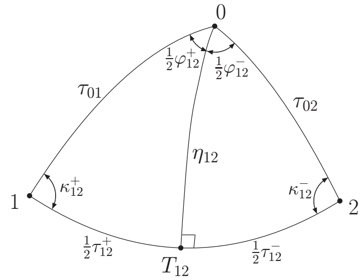

Figure 3. An asymmetric spherical triangle 012

$$\begin{aligned}
\cos\left(\tfrac{1}{2}\tau_{12}^{-}\right) &= \frac{\cos\tau_{02}}{\cos\eta_{12}}, \\
\cos\left(\tfrac{1}{2}\varphi_{12}^{+}\right) &= \frac{\tan\eta_{12}}{\tan\tau_{01}}, \\
\cos\left(\tfrac{1}{2}\varphi_{12}^{-}\right) &= \frac{\tan\eta_{12}}{\tan\tau_{02}}, \\
\sin\left(\tfrac{1}{2}\tau_{12}^{+}\right) &= \sin\tau_{01}\sin\left(\tfrac{1}{2}\varphi_{12}^{+}\right), \\
\sin\left(\tfrac{1}{2}\tau_{12}^{-}\right) &= \sin\tau_{02}\sin\left(\tfrac{1}{2}\varphi_{12}^{-}\right), \\
\tan\left(\tfrac{1}{2}\tau_{12}^{\pm}\right) &= \sin\eta_{12}\tan\left(\tfrac{1}{2}\varphi_{12}^{\pm}\right), \\
\sin\eta_{12} &= \sin\tau_{01}\sin\kappa_{12}^{+} = \sin\tau_{02}\sin\kappa_{12}^{-}, \\
\cos\kappa_{12}^{\pm} &= \cos\eta_{12}\sin\left(\tfrac{1}{2}\varphi_{12}^{\pm}\right).
\end{aligned}$$

Worth noting is

$$\cos\eta_{12} = \frac{m_0\sqrt{k_{12}^2}}{m_1 m_2 \sin\tau_{12}}. \tag{5}$$

Therefore,

$$\begin{aligned}
\cos\left(\tfrac{1}{2}\tau_{12}^{+}\right) &= \frac{m_2\sin\tau_{12}}{\sqrt{k_{12}^2}}, \\
\sin\left(\tfrac{1}{2}\tau_{12}^{+}\right) &= \frac{m_1^2 - m_2^2 + k_{12}^2}{2m_1\sqrt{k_{12}^2}}, \\
\cos\left(\tfrac{1}{2}\tau_{12}^{-}\right) &= \frac{m_1\sin\tau_{12}}{\sqrt{k_{12}^2}}, \\
\sin\left(\tfrac{1}{2}\tau_{12}^{-}\right) &= \frac{m_2^2 - m_1^2 + k_{12}^2}{2m_2\sqrt{k_{12}^2}}.
\end{aligned}$$

In a triangle with the sides m_1, m_2 and $\sqrt{k_{12}^2}$, the angles $\frac{1}{2}\tau_{12}^{+}$ and $\frac{1}{2}\tau_{12}^{-}$ are those between the height of the triangle and the sides m_1 and m_2, respectively. Therefore, the point T_{12} in Fig. 3 corresponds to the intersection of this face height and the sphere.

3. HYPERGEOMETRIC STUFF

Eqs. (3.38)–(3.39) of [3] yield

$$J^{(3)}(n;1,1,1) = -\frac{\mathrm{i}\pi^{n/2}\Gamma\left(3-\frac{n}{2}\right)}{m_0^{4-n}\sqrt{\Lambda^{(3)}}}\,\Omega^{(3;n)}, \tag{6}$$

where $\Omega^{(3;n)}$ is an integral over the solid angle $\Omega^{(3)}$ (corresponding to triangle 123 in Fig. 1b),

$$\Omega^{(3;n)} = \int\!\!\int\limits_{\Omega^{(3)}} \frac{\sin^{n-2}\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{\cos^{n-3}\theta}. \tag{7}$$

According to Fig. 2 and Fig. 3, $\Omega^{(3;n)}$ can be presented as a sum of six contributions:

$$\begin{aligned}\Omega^{(3;n)} &= \omega\left(\tfrac{1}{2}\varphi_{12}^{+},\eta_{12}\right)+\omega\left(\tfrac{1}{2}\varphi_{12}^{-},\eta_{12}\right)\\ &+\omega\left(\tfrac{1}{2}\varphi_{23}^{+},\eta_{23}\right)+\omega\left(\tfrac{1}{2}\varphi_{23}^{-},\eta_{23}\right)\\ &+\omega\left(\tfrac{1}{2}\varphi_{31}^{+},\eta_{31}\right)+\omega\left(\tfrac{1}{2}\varphi_{31}^{-},\eta_{31}\right), \quad (8)\end{aligned}$$

with (see Refs. [9,10,11])

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right)=\frac{1}{2\varepsilon}\int\limits_0^{\varphi/2}\mathrm{d}\phi\left[1-\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon}\right], (9)$$

where $\varepsilon=\frac{1}{2}(4-n)$. Defining $\tan\frac{\tau}{2}=\sin\eta\ \tan\frac{\varphi}{2}$, we can obtain another useful representation,

$$\int\limits_0^{\varphi/2}\mathrm{d}\phi\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} = \sin\eta\ \cos^{2\varepsilon}\eta\int\limits_0^{\tau/2}\frac{\mathrm{d}\psi\ \cos^{2\varepsilon}\psi}{1-\cos^2\eta\ \cos^2\psi}\ . \quad (10)$$

The remaining ϕ-integral in Eq. (9) can be calculated using a substitution $\phi=\arctan\left(\frac{\sqrt{u}}{\sin\eta}\right)$. The result can be presented in terms of Appell's hypergeometric function F_1,

$$\begin{aligned}\omega\left(\tfrac{1}{2}\varphi,\eta\right) &= \frac{1}{2\varepsilon}\left[\frac{\varphi}{2}-\tan\tfrac{\varphi}{2}\ \cos^{2\varepsilon}\eta\right.\\ &\left.\times F_1\left(\tfrac{1}{2},1,\varepsilon;\tfrac{3}{2}\middle|\ -\tan^2\tfrac{\varphi}{2},-\tan^2\tfrac{\tau}{2}\right)\right]. \quad (11)\end{aligned}$$

Moreover, using the transformation formula

$$\begin{aligned}F_1(a,b,b';c|x,y) &= (1-x)^{-b}(1-y)^{-b'}\\ &\times F_1\left(c-a,b,b';c\middle|\ \frac{x}{x-1},\frac{y}{y-1}\right),\end{aligned}$$

the result can be presented as

$$\begin{aligned}\omega\left(\tfrac{1}{2}\varphi,\eta\right) &= \frac{1}{2\varepsilon}\left[\frac{\varphi}{2}-\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\cos^{2\varepsilon}\tau_0\right.\\ &\left.\times F_1\left(1,1,\varepsilon;\tfrac{3}{2}\middle|\ \sin^2\tfrac{\varphi}{2},\ \sin^2\tfrac{\tau}{2}\right)\right], \quad (12)\end{aligned}$$

with $\cos\tau_0=\cos\eta\ \cos\frac{\tau}{2}$. Similar F_1 functions occurred in Refs. [12,13] (cf. also Ref. [14]).

An important formula (shift $\varepsilon\to 1+\varepsilon$, or $n\to n-2$) reads:

$$\begin{aligned}F_1\left(1,1,\varepsilon;\tfrac{3}{2}\middle|\ x,y\right) &= \frac{y}{x}\ {}_2F_1\left(1,1+\varepsilon;\tfrac{3}{2}\ \middle|\ y\right)\\ &+\left(1-\frac{y}{x}\right)F_1\left(1,1,1+\varepsilon;\tfrac{3}{2}\middle|\ x,y\right)\ . \quad (13)\end{aligned}$$

It can be supplemented by a Kummer relation:

$$\begin{aligned}&(1-2\varepsilon)\ {}_2F_1\left(1,\varepsilon;\tfrac{3}{2}\middle|\ y\right)\\ &\quad =1-2\varepsilon(1-z)\ {}_2F_1\left(1,1+\varepsilon;\tfrac{3}{2}\middle|\ y\right)\ . \quad (14)\end{aligned}$$

Each of the three triangles in Fig. 2 may be associated with a specific three-point function $J_i^{(3)}(n;1,1,1)$. According to Eq. (3.45) of [3], the result of such splitting reads

$$\begin{aligned}&J^{(3)}(n;1,1,1)\\ &\quad =\frac{m_1^2m_2^2m_3^2}{\Lambda^{(3)}}\sum_{i=1}^{3}\frac{F_i^{(3)}}{m_i^2}\ J_i^{(3)}(n;1,1,1), \quad (15)\end{aligned}$$

where $F_i^{(3)}=\frac{\partial}{\partial m_i^2}\left(m_i^2\ D^{(3)}\right)$ (see also in Ref. [4]). The geometrical meaning of $F_i^{(3)}$ was discussed in [3]. In particular,

$$m_2^2m_3^2\ F_1^{(3)}+m_1^2m_3^2\ F_2^{(3)}+m_1^2m_2^2\ F_3^{(3)}=\Lambda^{(3)}$$

means that the volume of the basic tetrahedron equals the sum of the volumes after splitting.

For each of the integrals $J_i^{(3)}$, only one two-point function appears in the reduction formulae. For instance, using Eqs. (12) and (13) we get

$$\begin{aligned}&(n-2)\pi^{-1}\ J_3^{(3)}(n+2;1,1,1)\\ &\quad =-2m_0^2\ J_3^{(3)}(n;1,1,1)-J^{(3)}(n;1,1,0)\ , (16)\end{aligned}$$

and similarly for $J_1^{(3)}$ and $J_2^{(3)}$. This yields a geometrical way to derive the recursion in n: just take Eq. (15), shift $n\to n+2$ and substitute Eq. (16). The result is

$$\begin{aligned}&J^{(3)}(n+2;1,1,1)\\ &= -\frac{\pi m_1^2m_2^2m_3^2}{(n-2)\Lambda^{(3)}}\left\{2D^{(3)}J_3^{(3)}(n;1,1,1)\right.\\ &+\frac{F_1^{(3)}}{m_1^2}\ J^{(3)}(n;0,1,1)+\frac{F_2^{(3)}}{m_2^2}\ J^{(3)}(n;1,0,1)\\ &\left.+\frac{F_3^{(3)}}{m_3^2}\ J^{(3)}(n;1,1,0)\right\}, \quad (17)\end{aligned}$$

in agreement with Refs. [15,13] (see also [16,17]).

4. SPECIAL VALUES OF n

4.1. $n=3$ $(\varepsilon=\frac{1}{2})$

In this case, we can use the known reduction formula for the F_1 function,

$$F_1\left(a,b,b';b+b'\middle|\ x,y\right)$$

$$= (1-y)^{-a}\ {}_2F_1\left(a, b; b+b' \Big|\ \frac{x-y}{1-y}\right) \ . \quad (18)$$

Taking into account that

$${}_2F_1\left(1,1;\tfrac{3}{2}\Big|\ z\right) = \frac{\arcsin\sqrt{z}}{\sqrt{z(1-z)}} \ ,$$

we get

$$F_1\left(1,1,\tfrac{1}{2};\tfrac{3}{2}\Big|\ \sin^2\tfrac{\varphi}{2},\ \sin^2\tfrac{\tau}{2}\right) = \frac{\pi-2\kappa}{\sin\varphi\ \cos\tau_0} \ ,$$

with $\cos\kappa = \sin\frac{\varphi}{2}\cos\eta$ and $\cos\tau_0 = \cos\frac{\tau}{2}\cos\eta$. Therefore,

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right)\Big|_{n=3} = \frac{\varphi}{2} - \frac{\pi}{2} + \kappa \ . \quad (19)$$

Collecting results for all six triangles, we reproduce Eq. (5.4) of [3] (see also in [4]),

$$\Omega^{(3;3)} = \Omega^{(3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi \ . \quad (20)$$

4.2. $n = 2$ **(**$\varepsilon = 1$**)**

In this case, the function F_1 reduces to

$$F_1\left(1,1,1;\tfrac{3}{2}\Big|\ x,y\right) = \frac{1}{x-y}\left[\frac{\sqrt{x}\ \arcsin\sqrt{x}}{\sqrt{1-x}} - \frac{\sqrt{y}\ \arcsin\sqrt{y}}{\sqrt{1-y}}\right] \ .$$

In this way, we get

$$F_1\left(1,1,1;\tfrac{3}{2}\Big|\ \sin^2\tfrac{\varphi}{2},\ \sin^2\tfrac{\tau}{2}\right) = \frac{\varphi - \tau\sin\eta}{\sin\varphi\ \cos^2\tau_0}$$

and, therefore,

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right)\Big|_{n=2} = \tfrac{1}{4}\tau\sin\eta \ . \quad (21)$$

Collecting results for all six triangles, we get

$$\Omega^{(3;2)} = \tfrac{1}{2}\left(\tau_{12}\sin\eta_{12} + \tau_{23}\sin\eta_{23} + \tau_{13}\sin\eta_{13}\right) \ .$$

Recalling that the two-point integral in two dimensions is proportional to $\tau/\sin\tau$ (see Eq. (4.3) of Ref. [3]), we see that the three-point integral (with $n = 2$) is a combination of three two-point integrals, with coefficients proportional to $\sin\tau_{jl}\sin\eta_{jl}$ (cf. Eq. (10) of Ref. [4]).

4.3. $n = 5$ **(**$\varepsilon = -\frac{1}{2}$**)**

In this case, we obtain

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right)\Big|_{n=5} = \frac{\pi}{2} - \frac{\varphi}{2} - \kappa + \frac{1}{2}\tan\eta\ \ln\left(\frac{1+\sin\frac{\tau}{2}}{1-\sin\frac{\tau}{2}}\right) .$$

Collecting results for all six triangles, we get

$$\begin{aligned}\Omega^{(3;5)} = &-\left(\psi_{12}+\psi_{23}+\psi_{31}-\pi\right) \\ &+\tan\eta_{12}\ \ln\left(\frac{m_1+m_2+\sqrt{k_{12}^2}}{m_1+m_2-\sqrt{k_{12}^2}}\right) \\ &+\tan\eta_{23}\ \ln\left(\frac{m_2+m_3+\sqrt{k_{23}^2}}{m_2+m_3-\sqrt{k_{23}^2}}\right) \\ &+\tan\eta_{13}\ \ln\left(\frac{m_1+m_3+\sqrt{k_{13}^2}}{m_1+m_3-\sqrt{k_{13}^2}}\right) .\end{aligned}$$

In other words, the five-dimensional three-point integral can be expressed in terms of the three-dimensional three- and two-point integrals (see Eqs. (5.4) and (4.6) of [3]), in agreement with Eq. (17).

4.4. $n = 4$ **(**$\varepsilon \to 0$**)**

In this case, we need to expand the F_1 function up to the term linear in ε (it is easy to see that the ε^0 term cancels the $\frac{\varphi}{2}$ contribution). The result can be presented in terms of Clausen function, see Eq. (5.21) of [3]. Collecting three contributions of this type, we get $6\times 3 = 18$ Clausen functions that can be analytically continued in terms of 12 dilogarithms [18]. In Ref. [6] the result is presented in terms of 15 Clausen functions.

5. ANALYTIC CONTINUATION

In the integral occurring in Eq. (9), let us substitute $z \Rightarrow e^{2i\phi}$, so that $\cos^2\phi \Rightarrow \frac{(1+z)^2}{4z}$ and

$$1 + \frac{\tan^2\eta}{\cos^2\phi} \Rightarrow \frac{(z+\rho)(z+1/\rho)}{(z+1)^2}, \quad (22)$$

with

$$\rho \equiv \frac{1-\sin\eta}{1+\sin\eta} \ . \quad (23)$$

In this way, we get

$$\int\limits_0^{\varphi/2} \mathrm{d}\phi\ \left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} \Rightarrow \frac{\mathrm{i}}{2}\int\limits_{z_0}^{1}\frac{\mathrm{d}z}{z}\ \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right]^{-\varepsilon} , \quad (24)$$

with $z_0 \leftrightarrow e^{\mathrm{i}\varphi}$. Expanding in ε, we get

$$Q_j \equiv \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z}\ \ln^j \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right] . \qquad (25)$$

The first term,

$$\begin{aligned} Q_1 &= \mathrm{Li}_2\left(-z_0\rho\right) + \mathrm{Li}_2\left(-z_0/\rho\right) - 2\mathrm{Li}_2\left(-z_0\right) \\ &\quad + \tfrac{1}{2}\ln^2\rho , \end{aligned} \qquad (26)$$

yields the known result [18] for the three-point function in four dimensions. The r.h.s. of Eq. (26) can also be presented as

$$\begin{aligned} &2\mathrm{Li}_2\left(z_0\right) - \mathrm{Li}_2\left(\frac{\rho+z_0}{\rho+z_0^{-1}}\right) - \mathrm{Li}_2\left(z_0^2\ \frac{\rho+z_0^{-1}}{\rho+z_0}\right) \\ &\quad + \tfrac{1}{2}\ln^2\rho - \tfrac{1}{2}\ln^2\left[\frac{z_0\rho(\rho+z_0^{-1})}{\rho+z_0}\right] , \end{aligned} \qquad (27)$$

in agreement with Eq. (5.17) of Ref. [3].

The second term, Q_2, gives the ε term of the three-point function. For $j=2$, the integral (25) can be evaluated in terms of polylogarithms,

$$Q_2 = Q_2^{(1)}(z_0,\rho)\ln\rho + Q_2^{(2)}(z_0,\rho) , \qquad (28)$$

$$\begin{aligned} Q_2^{(1)} &= 2\mathrm{Li}_2\left(\frac{1-\rho}{1+z_0\rho}\right) + 2\mathrm{Li}_2\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) \\ &\quad -2\mathrm{Li}_2\left(\frac{\rho-1}{z_0+\rho}\right) - 2\mathrm{Li}_2\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \\ &\quad -\mathrm{Li}_2\left(\frac{1-\rho^2}{1+z_0\rho}\right) - \mathrm{Li}_2\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) \\ &\quad +\mathrm{Li}_2\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{Li}_2\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right) , \\ Q_2^{(2)} &= 4\mathrm{S}_{1,2}\left(\frac{1-\rho}{1+z_0\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) \\ &\quad +4\mathrm{S}_{1,2}\left(\frac{\rho-1}{z_0+\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \\ &\quad -\mathrm{S}_{1,2}\left(\frac{1-\rho^2}{1+z_0\rho}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) \\ &\quad -\mathrm{S}_{1,2}\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right) . \end{aligned}$$

Note that the arguments of Nielsen polylogarithms $\mathrm{S}_{1,2}$ are the same as the arguments of Li_2. Furthermore,

$$\begin{aligned} \frac{\partial}{\partial\rho}\, Q_2^{(2)}(z_0,\rho) &= -\ln\rho\, \frac{\partial}{\partial\rho}\, Q_2^{(1)}(z_0,\rho) , \\ \rho\, \frac{\partial}{\partial\rho}\, Q_2(z_0,\rho) &= Q_2^{(1)}(z_0,\rho) . \end{aligned}$$

Another useful representation for $Q_2^{(1)}$ reads

$$\begin{aligned} Q_2^{(1)} &= 2\mathrm{Li}_2\left(-\frac{\rho+z_0}{1+\rho z_0}\right) - 2\mathrm{Li}_2\left(-\frac{1+\rho z_0}{\rho+z_0}\right) \\ &\quad +2\ln\left[\frac{\rho}{(\rho+1)^2}\right]\ln\left(\frac{1+\rho z_0}{\rho+z_0}\right) . \end{aligned} \qquad (29)$$

The occurring $\mathrm{S}_{1,2}$ functions can be presented in terms of trilogarithms Li_3. This result corresponds to Eq. (82) of Ref. [13]. We note that the first calculation of the ε-term of the one-loop three-point function was given in Ref. [19] (see also Ref. [20] for the off-shell massless case).

Eq. (25) shows that all higher terms of the ε-expansion of the one-loop three-point function can be expressed in terms of one-fold integrals of the products of logarithms of three linear arguments. (For some specific configurations, the ε^2 terms were studied in Ref. [21].) The considered representations may be useful to understand the types of generalized functions needed to decsribe the analytic structure of the results for higher terms of the ε-expansion.

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